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Strong Law of Large Numbers and Growth Rate for Some Dependent Sequences*

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Abstract: In this paper, we study the strong limit properties of the PA sequence, the NA sequence, the strongly positive dependent (SPD) sequence and the martingale difference sequence. By using the maximal moment inequalities of the NA sequence, the demimartingale sequence and the truncated method of random variables, we obtain the strong law of large numbers, the strong convergence rate and the integrability of supremum for the aforementioned dependent sequences. This study not only generalizes the strong law of large numbers for independent sequences to the case of dependent sequences, but also obtains their convergence rate.

Keywords: strong law of large numbers; strong growth rate; PA sequence; NA sequence

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1 Introduction

Let $X_1, X_2, \cdots, X_n$ denote a sequence of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$.

$$S_n = \sum_{i=1}^{n} X_i, \quad S_0 = 0, \quad X^+ = \max(0, X), \quad X^- = \max(0, -X),$$

and I_A is the indicator function of set A. Strong law of large numbers (SLLN) for random variables has often been concerned, for example, Yang *et al*[1] obtained the SLLN for positively associated (PA) sequence and negatively associated (NA) sequence. The main purpose of this paper is to establish the SLLN, strong growth rate and the integrability of supremum for some dependent sequences (such as PA sequence, NA sequence, strongly positive dependent (SPD) sequence and martingale difference sequence) by using the general method to prove the SLLN[2,3]. In addition, we give an example which satisfies the conditions of the proposed Theorem 2.1, Theorem 2.2 and Theorem 2.4 (for $\phi(x) = x^2$, $b_k = k$).

In many mathematic and mechanic models, assumptions of dependent random variables in the models are more reasonable than the case of independent random variables. Therefore, some dependent sequences have received more and more attention. For example, the concept

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of the PA sequence was introduced by Esary, Proscan and Walkup[4]. Newman and Wright[5] introduced the concept of demi (sub) martingle, and pointed out that the partial sums of a mean zero PA sequence is a demimartingale. It can be checked that a martingale is a demimartingale. Many authors have studied the properties about demimartingale. For example, Newman and Wright[5] extended the Doob's maximal inequality for martingale to the case of demimartingale; Christofides[6] extended the Chow's maximal inequality of submartingale to the case of demisubmartingale; Hu et al[3] studied the strong growth rate for martingale and demimartingale, etc. Joag-Dev and Proschan[7] introduced the concept of NA sequence, Matula[8] extended the three series theorem to this case, and Shao[9] obtained the moment inequalities for it. Zheng[10] gave the definition of SPD sequence and obtained some inequalities for it. Based on the inequality (3) in [10] and the definition of demimartingale, it is easy to see that the partial sums of a mean zero SPD sequence is a demimartingale. For other dependent sequences such as $\tilde{\rho}$-mixing sequence and $\tilde{\varphi}$-mixing sequence, the reader can refer to Utev and Peligrad[11], Wu[12] or Wang et al[13], respectively. Now, we give a lemma as follows:

Lemma 1.1 Let $\{X_i\}_{i\geq 1}$ be a sequence of random variables and $b_1, b_2, \cdots$ be a nondecreasing sequence of positive numbers. Suppose that $\phi : R \to R^+$ is an even function which is nondecreasing on $[0,\infty)$, and $x^p/\phi(x)$ is nondecreasing on $[0,\infty)$ for some $p > 0$. Then

$$E|X_i|^p \leq b_i^p \frac{E[\phi(X_i)I(|X_i| \leq b_i)]}{\phi(b_i)} + E\big[\,|X_i|^p I(|X_i| > b_i)\big], \quad i \geq 1.$$

Proof Under the conditions of Lemma 1.1, we can get that

$$\frac{|X_i|^p}{b_i^p} \leq \frac{\phi(X_i)}{\phi(b_i)}, \quad \text{for} \quad |X_i| \leq b_i.$$

Therefore

$$\begin{aligned} E|X_i|^p &= b_i^p E\big[(|X_i|^p/b_i^p)I(|X_i| \leq b_i)\big] + E\big[\,|X_i|^p I(|X_i| > b_i)\big] \\ &\leq b_i^p \frac{E[\phi(X_i)I(|X_i| \leq b_i)]}{\phi(b_i)} + E\big[\,|X_i|^p I(|X_i| > b_i)\big]. \end{aligned}$$

2 Main results

Theorem 2.1 Let $\{X_n\}_{n\geq 1}$ be a mean zero PA sequence and $b_1, b_2, \cdots$ be a nondecreasing unbounded sequence of positive numbers. Suppose that $\phi : R \to R^+$ is an even function which is nondecreasing on $[0,\infty)$, and $x^2/\phi(x)$ is nondecreasing on $[0,\infty)$. If

$$\sum_{k=1}^{\infty} \frac{\mathrm{Cov}(X_k, S_k)}{b_k^2} < \infty, \tag{1}$$

$$\sum_{k=1}^{\infty} \frac{E[\phi(X_k)I(|X_k| \leq b_k)]}{\phi(b_k)} < \infty, \tag{2}$$

then

$$\lim_{n\to\infty} \frac{S_n}{b_n} = 0, \quad \text{a.s.}, \tag{3}$$

with the growth rate

$$\frac{S_n}{b_n} = O\Big(\frac{\beta_n}{b_n}\Big), \quad \text{a.s.}, \tag{4}$$

where

$$\beta_n = \max_{1\le k\le n} b_k \nu_k^{\delta/2}, \quad \forall\, 0<\delta<1, \quad \nu_n = \sum_{k=n}^{\infty} \frac{\alpha_k}{b_k^2}, \quad \lim_{n\to\infty} \frac{\beta_n}{b_n} = 0, \tag{5}$$

$$\alpha_k = 8\mathrm{Cov}(X_k, S_{k-1}) + 4b_k^2 \frac{E[\phi(X_k)I(|X_k|\le b_k)]}{\phi(b_k)} + 4E\big[X_k^2 I\big(|X_k|>b_k\big)\big], \quad k\ge 1, \tag{6}$$

and for all $0<r<2$,

$$E\Big(\sup_{n\ge1}\Big|\frac{S_n}{b_n}\Big|^r\Big) \le 1 + \frac{2r}{2-r}\sum_{k=1}^{\infty}\Big\{\frac{E[\phi(X_k)I(|X_k|\le b_k)]}{\phi(b_k)} + \frac{E[X_k^2 I(|X_k|>b_k)]}{b_k^2} + 2\frac{\mathrm{Cov}(X_k,S_{k-1})}{b_k^2}\Big\} < \infty, \tag{7}$$

$$E\Big(\sup_{n\ge1}\Big(\frac{S_n}{b_n}\Big)^2\Big) \le 16\sum_{k=1}^{\infty}\Big\{2\frac{\mathrm{Cov}(X_k,S_{k-1})}{b_k^2} + \frac{E[\phi(X_k)I(|X_k|\le b_k)]}{\phi(b_k)} + \frac{E[X_k^2 I(|X_k|>b_k)]}{b_k^2}\Big\} < \infty. \tag{8}$$

Proof Since $\{X_n\}_{n\ge1}$ is a mean zero PA sequence, $\{S_n\}_{n\ge1}$ is a demimartingale. According to Corollary 2.1 and Lemma 2.1 of Christofides[6], $\{(S_n^+)^2\}_{n\ge1}$ and $\{(S_n^-)^2\}_{n\ge1}$ are demisubmartingales. So based on the definition of demisubmartingale, for $j=1,2,\cdots$, it has

$$E(S_{j+1}^+)^2 \ge E(S_j^+)^2, \quad E(S_{j+1}^-)^2 \ge E(S_j^-)^2,$$

which implies

$$E(S_{j+1}^2 - S_j^2) = E(S_{j+1}^+)^2 - E(S_j^+)^2 + E(S_{j+1}^-)^2 - E(S_j^-)^2 \ge 0.$$

Notice that

$$\alpha_1 = 4b_1^2\frac{E[\phi(X_1)I(|X_1|\le b_1)]}{\phi(b_1)} + 4E\big[X_1^2 I\big(|X_1|>b_1\big)\big] \ge 0,$$

$$E(X_kS_{k-1}) = \mathrm{Cov}(X_k, S_{k-1}).$$

Thus, based on (6) and Lemma 1.1, for $k\ge2$, it follows that

$$\begin{aligned} 0 &\le E(S_k^2 - S_{k-1}^2) = E\big\{(S_k - S_{k-1})(S_k + S_{k-1})\big\} \\ &= E\big\{X_k(2S_{k-1}+X_k)\big\} = 2E(X_kS_{k-1}) + EX_k^2 \\ &\le 2\mathrm{Cov}(X_k,S_{k-1}) + b_k^2\frac{E[\phi(X_k)I(|X_k|\le b_k)]}{\phi(b_k)} + E\big[X_k^2 I\big(|X_k|>b_k\big)\big] \\ &= \frac{1}{4}\alpha_k. \end{aligned} \tag{9}$$

Next, by virtue of (1), (2), (9) and Lemma 1.4 (with $p=2$) of [3], we obtain that

$$E\big(\max_{1\le l\le n}|S_l|\big)^2 \le 4ES_n^2 = 4\sum_{k=1}^{n} E(S_k^2 - S_{k-1}^2) \le \sum_{k=1}^{n}\alpha_k, \quad n\ge 1, \tag{10}$$

$$\sum_{k=1}^{\infty}\frac{\alpha_k}{b_k^2} = 4\sum_{k=1}^{\infty}\Big\{2\frac{\mathrm{Cov}(X_k,S_{k-1})}{b_k^2} + \frac{E[\phi(X_k)I(|X_k|\le b_k)]}{\phi(b_k)} + \frac{E[X_k^2 I(|X_k|>b_k)]}{b_k^2}\Big\} < \infty. \tag{11}$$

Therefore, (3)-(6) and (8) follow Lemma 1.5 of [3].

Now, we prove (7). For all $0<r<2$, $t>0$, by (1), (2), Lemma 1.1 and Theorem 2.2 of [14], it has

$$\begin{aligned}
&P\Big(\sup_{n\ge1}\Big|\frac{S_n}{b_n}\Big|^r > t\Big)\\
&\le \lim_{N\to\infty} P\Big(\max_{1\le n\le N}\Big|\frac{S_n}{b_n}\Big| > t^{1/r}\Big) \le \lim_{N\to\infty}\frac{2}{t^{2/r}}\Big\{\sum_{k=1}^{N}\frac{EX_k^2}{b_k^2} + 2\sum_{k=1}^{N}\frac{\mathrm{Cov}(X_k,S_{k-1})}{b_k^2}\Big\}\\
&\le \frac{2}{t^{2/r}}\sum_{k=1}^{\infty}\Big\{\frac{E[\phi(X_k)I(|X_k|\le b_k)]}{\phi(b_k)} + \frac{E[X_k^2 I(|X_k|>b_k)]}{b_k^2} + 2\frac{\mathrm{Cov}(X_k,S_{k-1})}{b_k^2}\Big\},
\end{aligned}$$

which implies that

$$\begin{aligned}
&E\Big(\sup_{n\ge1}\Big|\frac{S_n}{b_n}\Big|^r\Big) = \int_0^{\infty} P\Big(\sup_{n\ge1}\Big|\frac{S_n}{b_n}\Big|^r > t\Big)\mathrm{d}t\\
&= \int_0^1 P\Big(\sup_{n\ge1}\Big|\frac{S_n}{b_n}\Big|^r > t\Big)\mathrm{d}t + \int_1^{\infty} P\Big(\sup_{n\ge1}\Big|\frac{S_n}{b_n}\Big|^r > t\Big)\mathrm{d}t\\
&\le 1 + 2\sum_{k=1}^{\infty}\Big\{\frac{E[\phi(X_k)I(|X_k|\le b_k)]}{\phi(b_k)} + \frac{E[X_k^2 I(|X_k|>b_k)]}{b_k^2} + 2\frac{\mathrm{Cov}(X_k,S_{k-1})}{b_k^2}\Big\}\int_1^{\infty} t^{-2/r}\mathrm{d}t\\
&= 1 + \frac{2r}{2-r}\sum_{k=1}^{\infty}\Big\{\frac{E[\phi(X_k)I(|X_k|\le b_k)]}{\phi(b_k)} + \frac{E[X_k^2 I(|X_k|>b_k)]}{b_k^2} + 2\frac{\mathrm{Cov}(X_k,S_{k-1})}{b_k^2}\Big\} < \infty.
\end{aligned}$$

Remark 2.1 In the Theorem 3.1 of [1], they got the strong law of large numbers (SLLN) for PA sequence. In this paper, we obtain the SLLN, strong growth rate and the integrability of supremum for PA sequence. Let $\phi(x)=|x|^p$ in Lemma 1.1, we can get that

$$E|X_k|^p \le b_k^p\frac{E[\phi(X_k)I(|X_k|\le b_k)]}{\phi(b_k)} + E\big[|X_k|^p I\big(|X_k|>b_k\big)\big] = E|X_k|^p.$$

In addition, if the condition (1) holds, then for $\phi(x)=x^2$,

$$\sum_{k=1}^{\infty}\frac{E[\phi(X_k)I(|X_k|\le b_k)]}{\phi(b_k)} = \sum_{k=1}^{\infty}\frac{E[X_k^2 I(|X_k|\le b_k)]}{b_k^2} < \infty.$$

Hence, by taking $\phi(x) = x^2$ in Theorem 2.1, we can obtain Theorem 3.1 of Hu *et al*[14] immediately.

Theorem 2.2 Let $\{X_n\}_{n\geq 1}$ be a mean zero SPD sequence, and $b_1, b_2, \cdots$ be a nondecreasing unbounded sequence of positive numbers. Suppose that $\phi : R \to R^+$ is an even function which is nondecreasing on $[0, \infty)$, and $x^2/\phi(x)$ is nondecreasing on $[0, \infty)$. Assume further that (1) and (2) are satisfied, then (3)-(8) hold true.

Proof By inequality (2) in Zheng[10], we have $\mathrm{Cov}(X_j, X_k) = EX_jX_k - EX_jEX_k \geq 0$ for all j, $k \geq 1$. Since $EX_n = 0$, by inequality (3) in Zheng[10], for all $n \geq 1$, we can see that

$$\begin{aligned} & E\{f(S_1, \cdots, S_n)(S_{n+1} - S_n)\} \\ = \; & E\{f(X_1, X_1 + X_2, \cdots, X_1 + X_2 + \cdots + X_n)X_{n+1}\} \geq 0 \end{aligned}$$

for all coordinatewise nondecreasing functions f such that the expectation is defined. So $\{S_n\}_{n\geq 1}$ is a demimartingale. Similar to the proof of Theorem 2.2 of Hu *et al*[14], we can obtain the Hájek-Rényi-type inequalities (2.4) and (2.5) in Hu *et al*[14] for the mean zero SPD sequence with $EX_n^2 < \infty$, $n \geq 1$. Thus, we can get the results of Theorem 2.2 immediately from the proof of Theorem 2.1.

Theorem 2.3 Let $\{X_n\}_{n\geq 1}$ be a mean zero NA sequence with $E|X_n|^p < \infty$, $n \geq 1$, $1 \leq p \leq 2$, and $b_1, b_2, \cdots$ be a nondecreasing unbounded sequence of positive numbers. Assume that $\phi : R \to R^+$ is an even function which is nondecreasing on $[0, \infty)$, and $x^p/\phi(x)$ is nondecreasing on $[0, \infty)$. If

$$\sum_{k=1}^{\infty} \frac{E[|X_k|^p I(|X_k| > b_k)]}{b_k^p} < \infty, \tag{12}$$

and (2) is satisfied, then (3) and (4) hold true, where

$$\beta_n = \max_{1\leq k\leq n} b_k \nu_k^{\delta/p}, \quad \forall\, 0 < \delta < 1, \quad \lim_{n\to\infty} \frac{\beta_n}{b_n} = 0, \tag{13}$$

$$\nu_n = 2^{3-p} \sum_{k=n}^{\infty} \Big\{ \frac{E[\phi(X_k) I(|X_k| \leq b_k)]}{\phi(b_k)} + \frac{E[|X_k|^p I(|X_k| > b_k)]}{b_k^p} \Big\}, \tag{14}$$

and

$$E\Big(\sup_{n\geq 1} \Big|\frac{S_n}{b_n}\Big|\Big)^p \leq 2^{5-p} \sum_{k=1}^{\infty} \Big\{ \frac{E[\phi(X_k) I(|X_k| \leq b_k)]}{\phi(b_k)} + \frac{E[|X_k|^p I(|X_k| > b_k)]}{b_k^p} \Big\} < \infty. \tag{15}$$

Proof Since

$$\alpha_k := 2^{3-p} \Big\{ b_k^p \frac{E[\phi(X_k) I(|X_k| \leq b_k)]}{\phi(b_k)} + E\big[|X_k|^p I(|X_k| > b_k)\big] \Big\} \geq 0,$$

by (2), (12), the inequality (1.6) in Theorem 2 (for $1 \leq p \leq 2$) of Shao[9] and Lemma 1.1, we have

$$E\big(\max_{1\leq l\leq n} |S_l|\big)^p \leq 2^{3-p} \sum_{k=1}^{n} E|X_k|^p \leq \sum_{k=1}^{n} \alpha_k, \quad n \geq 1, \tag{16}$$

$$\sum_{k=1}^{\infty} \frac{\alpha_k}{b_k^p} = 2^{3-p} \sum_{k=1}^{\infty} \Big\{ \frac{E[\phi(X_k) I(|X_k| \leq b_k)]}{\phi(b_k)} + \frac{E[|X_k|^p I(|X_k| > b_k)]}{b_k^p} \Big\} < \infty. \tag{17}$$

Hence, by Lemma 1.5 of [3], we can get (3), (4) and (13)-(15) immediately.

Corollary 2.1 Let $\{X_n\}_{n\geq 1}$ be a mean zero NA sequence. If

$$\sum_{k=1}^{\infty} \frac{EX_k^2}{b_k^2} < \infty, \tag{18}$$

then (3) and (4) hold true, where

$$\beta_n = \max_{1\leq k\leq n} b_k \nu_k^{\delta/2}, \quad \forall\, 0<\delta<1, \quad \lim_{n\to\infty} \frac{\beta_n}{b_n} = 0, \quad \nu_n = 2\sum_{k=n}^{\infty} \frac{EX_k^2}{b_k^2}, \tag{19}$$

and

$$E\Big(\sup_{n\geq 1}\Big|\frac{S_n}{b_n}\Big|^r\Big) \leq 1 + \frac{8r}{2-r}\sum_{k=1}^{\infty}\frac{EX_k^2}{b_k^2} < \infty, \quad \text{for} \quad 0<r<2, \tag{20}$$

$$E\Big(\sup_{n\geq 1}\Big(\frac{S_n}{b_n}\Big)^2\Big) \leq 8\sum_{k=1}^{\infty}\frac{EX_k^2}{b_k^2} < \infty. \tag{21}$$

Proof By Remark 2.1, we can find that conditions (2) and (12) are equivalent to the condition (18) if $\phi(x) = x^2$ and $p = 2$. Then by Theorem 2.3, it is easy to get (3), (4) and (21). Next, by (18) and the inequality (3.9) of [15], for all $0 < r < 2$, we can obtain (20) immediately from the proof of (7). The proof is omitted here.

Remark 2.2 In Theorem 4.1 of [1], they got the strong law of large numbers for NA sequence. Here, we obtain the strong law of large numbers, strong growth rate and integrability of supremum for NA sequence. Similarly, by Remark 2.1, let $\phi(x) = |x|^p$ in Theorem 2.3, we can get Theorem 3.1 of [2], where the conditions (2) and (12) are equivalent to

$$\sum_{k=1}^{\infty}\frac{E|X_k|^p}{b_k^p} < \infty.$$

Theorem 2.4 Let $\{X_n\}_{n\geq 1}$ be a martingale difference with respect to the filtration $\{\mathcal{F}_n\}_{n\geq 1}$, where the σ-field $\mathcal{F}_n$ is generated by random variables $X_1, X_2, \cdots, X_n$. Suppose that $EX_n^2 < \infty$ for each n. Let $b_1, b_2, \cdots$ be a nondecreasing unbounded sequence of positive numbers, and $\phi : R \to R^+$ be an even function which is nondecreasing on $[0,\infty)$, and $x^2/\phi(x)$ is nondecreasing on $[0,\infty)$. If

$$\sum_{k=1}^{\infty}\frac{E[X_k^2 I(|X_k| > b_k)]}{b_k^2} < \infty, \tag{22}$$

and (2) is satisfied, then (3) and (4) hold true, where

$$\beta_n = \max_{1\leq k\leq n} b_k \nu_k^{\delta/2}, \quad \forall\, 0<\delta<1, \quad \nu_n = \sum_{k=n}^{\infty}\frac{\alpha_k}{b_k^2}, \quad \lim_{n\to\infty}\frac{\beta_n}{b_n} = 0, \tag{23}$$

$$\alpha_k = 4b_k^2\frac{E[\phi(X_k)I(|X_k|\leq b_k)]}{\phi(b_k)} + 4E\big[X_k^2 I\big(|X_k| > b_k\big)\big], \quad k\geq 1, \tag{24}$$

and for all $0 < r < 2$,

$$E\Big(\sup_{n\geq 1}\Big|\frac{S_n}{b_n}\Big|^r\Big) \leq 1 + \frac{8r}{2-r}\sum_{k=1}^{\infty}\Big\{\frac{E[\phi(X_k)I(|X_k|\leq b_k)]}{\phi(b_k)} + \frac{E[X_k^2 I(|X_k|>b_k)]}{b_k^2}\Big\} < \infty, \tag{25}$$

$$E\Big(\sup_{n\geq 1}\Big(\frac{S_n}{b_n}\Big)^2\Big) \leq 16\sum_{k=1}^{\infty}\Big\{\frac{E[\phi(X_k)I(|X_k|\leq b_k)]}{\phi(b_k)} + \frac{E[X_k^2 I(|X_k|>b_k)]}{b_k^2}\Big\} < \infty. \tag{26}$$

Proof It is known that if $\{X_n\}_{n\geq 1}$ is a martingale difference sequence, then $\{S_n\}_{n\geq 1}$ is a martingale and a demimartingale. Notice that

$$\alpha_k = 4b_k^2\frac{E[\phi(X_k)I(|X_k|\leq b_k)]}{\phi(b_k)} + 4E\big[X_k^2 I(|X_k|>b_k)\big] \geq 0,$$

and for $j<k$, $E(X_jX_k) = E[X_jE(X_k|\mathcal{F}_{k-1})] = 0$. Then by (2), (22), Lemma 1.4 (with $p=2$) of [3] and Lemma 1.1, for $n\geq 1$, it follows

$$\begin{aligned} E\big(\max_{1\leq l\leq n}|S_l|\big)^2 &\leq 4ES_n^2 = 4\sum_{k=1}^{n}E(S_k^2 - S_{k-1}^2) \\ &= 4\sum_{k=1}^{n}\{2E(X_kS_{k-1}) + EX_k^2\} \leq \sum_{k=1}^{n}\alpha_k, \end{aligned} \tag{27}$$

$$\sum_{k=1}^{\infty}\frac{\alpha_k}{b_k^2} = 4\sum_{k=1}^{\infty}\Big\{\frac{E[\phi(X_k)I(|X_k|\leq b_k)]}{\phi(b_k)} + \frac{E[X_k^2 I(|X_k|>b_k)]}{b_k^2}\Big\} < \infty. \tag{28}$$

Therefore, (3), (4), (23), (24) and (26) follow from Lemma 1.5 of [3]. Next, we prove (25). It is easy to see that $\{\sum_{k=1}^{n}\frac{X_k}{b_k}\}_{n\geq 1}$ is a martingale sequence and a demimartingale sequence. Since $\{b_n\}_{n\geq 1}$ is a nondecreasing sequence of positive numbers, then for all $\varepsilon>0$, it has

$$\begin{aligned} \Big(\max_{1\leq k\leq n}\Big|\frac{S_k}{b_k}\Big|\geq\varepsilon\Big) &\subset \Big(\max_{1\leq k\leq n}\max_{1\leq i\leq k}\Big|\sum_{j=i}^{k}\frac{X_j}{b_j}\Big|\geq\varepsilon\Big) \\ &= \Big(\max_{1\leq i\leq k\leq n}\Big|\sum_{j=1}^{k}\frac{X_j}{b_j} - \sum_{j=1}^{i-1}\frac{X_j}{b_j}\Big|\geq\varepsilon\Big) \subset \Big(\max_{1\leq i\leq n}\Big|\sum_{j=1}^{i}\frac{X_j}{b_j}\Big|\geq\varepsilon/2\Big), \end{aligned}$$

(or see p228-229 in [15]). Hence, by Corollary 2.1 (with $\nu=2$) of [14] and Lemma 1.1, for all $0<r<2$, we can obtain (25) immediately from the proof of (7). The details are omitted here.

It can be seen that the conditions (2) and (22) are equivalent to the condition (18) if $\phi(x)=x^2$. Thus, we have the following corollary.

Corollary 2.2 Let $\{X_n\}_{n\geq 1}$ be a martingale difference with respect to the filtration $\{\mathcal{F}_n\}_{n\geq 1}$, where the σ-field $\mathcal{F}_n$ is generated by random variables $X_1, X_2, \cdots, X_n$, and $b_1, b_2, \cdots$ be a nondecreasing unbounded sequence of positive numbers. If (18) is satisfied, then (3), (4) and (20) hold true, where

$$\beta_n = \max_{1\leq k\leq n} b_k\nu_k^{\delta/2}, \quad \forall\, 0<\delta<1, \quad \lim_{n\to\infty}\frac{\beta_n}{b_n} = 0, \quad \nu_n = 4\sum_{k=n}^{\infty}\frac{EX_k^2}{b_k^2}, \tag{29}$$

and

$$E\Big(\sup_{n\geq 1}\Big(\frac{S_n}{b_n}\Big)^2\Big) \leq 16\sum_{k=1}^{\infty}\frac{EX_k^2}{b_k^2} < \infty. \tag{30}$$

Remark 2.3 Now, we give an example which satisfies the condition of Theorem 2.1, Theorem 2.2 and Theorem 2.4 (for $\phi(x) = x^2$ and $b_k = k$, $k \geq 1$). Consider a sequence of independent random variables $\{X_n\}$ such that for all $\alpha > 0$ and $n > n_0$,

$$P(X_n = n) = P(X_n = -n) = \frac{1}{2n(\log n)^{1+\alpha}}, \quad P(X_n = 0) = 1 - \frac{1}{n(\log n)^{1+\alpha}},$$

for $n \leq n_0$,

$$P(X_n = 1) = P(X_n = -1) = \frac{1}{2}.$$

It is easy to check that $EX_n = 0$ for all n and

$$DX_n = 1 \quad \text{if} \quad n \leq n_0, \quad DX_n = \frac{n}{(\log n)^{1+\alpha}} \quad \text{if} \quad n > n_0.$$

Let $b_k = k$, $k \geq 1$, then

$$\sum_{k=1}^{\infty}\frac{EX_k^2}{b_k^2} = \sum_{k=1}^{n_0}\frac{1}{k^2} + \sum_{k=n_0+1}^{\infty}\frac{1}{k(\log k)^{1+\alpha}} < \infty.$$

Remark 2.4 In the Corollary 2.2, we have a smaller factor 16 than the factor 452 (with $q = 1$) in (2.9) of Theorem 2.2 in [3]. In addition, we get the integrability of supremum for some dependent random sequences, such as (7), (8), (15), (20), (25), (26), (30).

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一些相依序列的强大数律和收敛速度

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摘 要: 本文研究了正相协序列、负相协序列、强正相依序列以及鞅差序列的强极限性质．利用负相协序列和弱鞅序列的极大值矩不等式以及随机变量的截尾方法，得到了上述相依序列的强大数定律、强收敛速度以及相应的随机变量序列上确界的可积性．本文不仅将独立情形下的强大数定律推广到以上相依序列，并且还给出了其收敛速度．

关键词: 强大数律；强收敛速度；正相协序列；负相协序列